THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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CONTENTS

	PAGE
MATHEMATICS AND MORALS. E. CUNNINGHAM, M.A.,	265
A QUESTION OF TIME. D. B. MAIR, M.A.,	271
ELEMENTARY METHODS OF TREATMENT OF CIRCULAR, HYPERBOLIC AND ELLIPTIC FUNCTIONS. W. MILLER, D.Sc., PH.D.,	275
HOMOGENEITY APPLIED TO ANGLES. PROF. D. M. Y. SOMMERVILLE, D.Sc.,	281
MATHEMATICAL NOTES (864–865). MAJOR C. H. CHEPMELL, R.A.; N. M. GIBBINS, M.A.,	284
REVIEWS. A. C. AITKEN, D.Sc.; E. R. HAMILTON, M.A.; PROF. W. H. TURNBULL, M.A.; PROF. G. N. WATSON, F.R.S.,	287
GLEANINGS FAR AND NEAR (408-418),	270
THE LIBRARY,	292
ERRATA,	292

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VOL. XIII. JANUARY, 1927. No. 186.

MATHEMATICS AND MORALS.*

BY E. CUNNINGHAM, M.A.

THE title of this paper was chosen before I had put my thoughts together, and even before I had considered thoroughly whether it were possible to make any general conclusions at all. I hope I shall not be accused afterwards of making something out of nothing, or of seeing connections where none exist. Perhaps it will be that, instead of having a vague notion that mathematics and morals are strangers or opposites, we may arrive at the end of the evening with a reasoned conviction that they are indeed so. But that the subject is not unprofitable to discuss I am strengthened in believing by noticing that Henri Poincaré, perhaps the greatest mathematician of the last fifty years, in the closing years of his life wrote an essay on the "Relations of Morals to Science," which is included in the volume of his last thoughts.

For Poincaré Science means Natural Philosophy, the ordered body of law found to hold in material phenomena. This body of law is at once the main field of application of mathematics and the prime source of pure mathematics. This last statement is worth amplifying a little, as it is very relevant to the main theme of this paper. I say that the main stimulus to and directing agency in the development of pure mathematics has been the attempt to express accurately and comprehensively, and with a minimum of assumptions, the ordered sequences of natural phenomena. I will go no further by way of illustration than to remark that in the course of his attempt to round off the work of the star-gazers from the ancient Greeks and Egyptians to his immediate predecessor Galileo, Sir Isaac Newton invented the differential and integral calculus, and discovered the Binomial Theorem. He little thought of the vile uses to which that theorem would be put as a scarifier of brains, and a plaything of examination-mongers.

I should like, in an oft-quoted passage, to remind you of Newton's assessment of his own work at the close of his life. "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

* Paper read to the Adams Society, St. John's College, Cambridge, Jan. 1926.

Here we have what seems to me to be the first close point of contact between Mathematics and Morals. If one of the ends of our life be the pursuit of beauty, then mathematics, properly understood, is one of the avenues we should follow.

Let us go to the other end of the scale from Newton's ripe achievements, which seemed to him to be but fleeting glimpses of the beauty behind all things. I remember the occasion when I happened to introduce my small daughter to the possibility of the magic square. After that she spent hours working out square after square, becoming, as it seemed to me, more and more convinced that there was some principle behind, quite unknown to her, which guarantees that it would come right. The mysterious order which becomes apparent in the squares of the pattern captivates for no reason that can strictly be analysed. The carrying out of the operation of filling up the square seems to be purely aesthetic.

Or to come back to the summits of human achievement again, this new work of Einstein, what is it that mainly gives it its value? It is not that "light has been caught bending" (*Daily Mail*), but that a harmony more all-embracing than any before dreamed of has been found to rule among the diverse phenomena of the natural world. Great as Newton's generalization had been, Einstein's has gone farther. For though men could not deny the tremendous successes of Newton's system of mechanics, though Laplace acclaimed it as final, yet there remained questionings. Philosophers boggled at his postulates of an absolute time and an absolute space; and Einstein, setting out to satisfy these malcontents, not only did so, but in doing so evolved a theory which included all that Newton had done, and completed it at the only points at which it had failed to be satisfactory.

I do not know if it would be agreed that harmony is the most essential factor in beauty, but it does seem to me that the desire for harmony animates the modern searcher after the ordered relations of the universe, as well as the pure mathematician spinning out of his own head, just as it animated the Greeks in their star-gazing and their geometry.

I find it very hard to distinguish the passion for truth from the quest of beauty. We search for truth with the heart as well as with the head. We follow it because we must. And we follow beauty with the head as well as with the heart. How else can we explain that traditional association of mathematics with music. I cannot tell for the life of me whether it is truth or beauty that reigns in a Bach fugue. Is it relentless logic or the beauty of the woven pattern that carry along at once both the head and the whole nervous being?

And so, while it would commonly be said that truth, and truth only, is the mathematician's sphere, I would assert that the appreciation of truth is indistinguishable from that of beauty, and that nothing can be beautiful that is not true and harmonious. And in saying this I do not fall back on a vague and sentimental use of the word true meaning merely satisfying to an unanalytic instinct. I mean, to take two illustrations, that a piece of statuary cannot be beautiful if its poise does not fit with the mathematical principles of mechanics, that a thought cannot be beautiful which is illogical.

But you will be protesting that, after all, the main sphere of morals is neither truth nor beauty, but goodness, and there you will at least say that mathematics is irrelevant.

You will say that, while mathematics may be the chief avenue to the perception of beauty and truth in the realm of material phenomena, when it comes to those personal relations in which intuition, spiritual receptivity and responsiveness are the characteristic qualities, the perception of number relations and of the indubitable laws of the material world are irrelevant or even misleading.

Yet I doubt if we should so hastily come to this conclusion. For it is

certain that our standards of personal goodness are not uniform and unchangeable. They are and must be in some respects relative to our understanding of the inevitable factors in human relationship. We cannot separate the sources of action which the spirit of good prompts from our conception of what is likely to follow from them. Here the mathematical faculty cannot fail to come into play. Intuition and Reason must go hand in hand.

I will only give here one illustration of my point, though it is a very comprehensive one. The economic problem of society consists in the control of the independent desires and motives of individuals by a recognition of the mathematical necessities and possibilities of their life together. On the one hand, we have the cry for the things which seem desirable. Labour's moral demand for a reasonable standard of life is unassailable. It has every moral right not only to a minimum wage, but to a maximum wage.

But what that maximum wage may be is determined to some extent by mathematical laws. Given the utmost good-will, and a perfected organization for securing that each worker shall put forth his maximum effort, yet the limitations of the accessibility of raw material, the depth of the coal-seam, the fertility of the soil are mathematical factors which set an upper bound to that maximum wage. Any moral demand for something that is mathematically not possible stultifies itself, produces controversy and misunderstanding, cynicism and despair. Only a person who is automatically correct in his mathematical appreciation of the inevitable relations involved in economics can properly give their due weight to the moral factors. That is why the economics of Prof. Marshall, the Senior Wrangler, are so human, while those of his non-mathematical disciples are less so.

It will be apparent from what I have said that it is necessary to draw a distinction between everyday mathematics and professional mathematics; between the calculations of the market-place and those of the Mathematical Tripos, or the learned societies; between the order the common man may appreciate in the world around him, and the latest mathematical treatise upon Relativity or Russell and Whitehead's *Principia Mathematica*.

I am not so sure that a similar distinction is not worth making in the sphere of morals between the plain duties and high impulses of which the average mortal is conscious, and the metaphysical arguments of the professors, which leave us uncertain whether our emotions of love, pity and compassion are realities or will-o'-the-wisps. For my part, I am content to leave the professors to their imaginings and to find the true significance of both mathematics and morals in the world of the common man's experience. Their sphere is not in lecture room and pulpit alone or chiefly, but everywhere and always and for everybody.

A nice doctrine, you will say, for a Cambridge Mathematical Teacher to promulgate, belittling the labours of his more learned colleagues under their very noses. Yet he who is most aloft and breathes the most pure air of the highest summits of abstraction will proclaim truth as his god and the sense of its beauty as his spur. The glory of truth is not quantity but quality, and in the oneness of truth the humblest beginners have their share.

You may agree with me or you may not when I say this in regard to ethical matters, but few of you will agree in regard of mathematics. You will say, remembering your school-days, either that it was the most disagreeable portion of your intellectual fare, or, if you were of the favoured few to whom it was palatable, that it was good fun for those who could do it and that is all. Moral significance it had none, interest and beauty only for the elect.

Now it seems to me that the conception of mathematics as a subject to be learned, rules to be applied, arguments to be memorized, is one of the greatest of the many evils that are being brought upon us by this terrific network of examinations in which our schools have been entangled through the mistaken enthusiasms of our University Local Examinations Syndicates and like

bodies. These bar the way of the teacher to whom has come the sacred calling of inspiring in youth the love of truth and beauty, who would lead them to shun dishonesty as a plague, and to find in the music of the spheres some echo of eternal realities.

Our children must at all costs be forced into the mould of these examinations. They must get through somehow. The chief aim of their mathematics is to get the answer. Their teachers complain if a question is set which does not conform to a routine type which can be dealt with according to well-practised methods. Facility in manipulation is the ideal which, all too surely, becomes the requirement of the wholesale examination. In the old days we had to be able to reproduce in every detail the verbal arguments of Euclid, to calculate correctly by logarithms, to reproduce the binomial theorm with a demonstration known to be incomplete by the writer of the text-book, but inserted as good enough for school-boys.

The moral effect was and is still disastrous—we were out to get the answer, never mind what it meant, to satisfy the teacher by something regardless of whether we understood or not.

And yet, if there had been time and freedom these same studies might have been of the highest moral value. That despised Euclid, now thrown overboard for modern hotch-potches, will stand for ever as a model of honest thinking. Whatever else Euclid may have been, he set us an immortal example of patient analysis of our rapidly formed judgments, sifting and sorting, tracking down the things which are taken for granted as axiomatic, and discovering the logical sequence of our own thought. No man who has once appreciated the organic unity of Euclid's work can again be content with a loose argument from undefined premises. If, further, he can bring his imagination to realize the work of a man great enough to sort out from the overpowering complexity of life this simple connected piece of truth, he cannot fail to feel that he is like a moth fluttering round the candle of life, singeing his wings badly at times in the heat, not knowing anything of the principles that underlie his existence.

I suggest, then, that the main function of mathematical teaching should be to present so much and no more of the great body of learning as can be appreciated, as part of the harmony of the universe and a model of the highest perfection in human thinking.

At this point you will, of course, begin to ask me to outline a syllabus and a technique. You will ask what should be the material, and how should it be presented? I must reply that that depends upon the teacher. It is more a question of enthusiasm and reverence than of syllabus. But I will just make one suggestion for a possible plan, not for beginners but for a later stage in school life. Let us go back to the Greeks, and begin with geometry and the stars. Let us map out, preferably practically, the paths of the planets, night by night. Let us rediscover Ptolemy's epicycles. Simultaneously let us learn something of the curves they discovered and called conic sections. Then let us come down in history to Copernicus and Kepler, and note how the patience and insight of the mathematician led to the discovery that the sun was the focus round which all the planets revolved in conic sections. Let us learn how Newton sought for an explanation of this, and, after having, quite as side-issues, discovered the binomial theorem and invented the infinitesimal calculus, gave to the world that immortal generalization, the law of gravitation, and a system of mechanics which enabled Laplace to say, in a moment of exaltation, "We are within sight of including under a single system of truth all the systems of the world."

It would not be impossible for the teacher round a programme of this kind to maintain a sense of unity with the world, of the wonder of the human intelligence, as well as of the majesty of the universe, and at the same time to instil the fundamental idea of honest accurate attention to detail, discontent with sloppy thinking and unfinished calculations. I would go further and

say, that a man's moral being is incomplete who has not felt these things, just as it is if he is content to speak slang instead of the full richness of his mother tongue.

I want to carry this a little further, and to put a question or two to those who are more familiar with psychology than I am. I want to know something about the psychology of dishonest learning. I want to know whether it is true that in our present way of teaching we do not seriously cause a suppression of the wish to understand for the sake of getting there in the competition.

Let me give an illustration. I once happened to be brought into close relations with a class of pupils who were learning logarithms. The teacher had given them a preliminary lesson on the theory. They were then asked to work out practical examples, multiplying numbers by the use of tables of logarithms. The remarks I heard showed me clearly that the theory had not been assimilated, and they were therefore groping around and guessing what they had to do. Their main desire was to get the right answer. It was like fitting a puzzle together. Given two numbers to multiply, and the answer, what way could they invent that would give the answer. They acquiesced in a fatalistic manner in the thought that the theory was beyond them, and concentrated on trying to find for themselves by chance the practical method.

Now, as I understand their minds, the teacher began by awakening an interest and opening a vista. The fact that $a^x a^y = a^{x+y}$ is, I believe, just one of those relations which do make a primary appeal, and give a feeling of universal truth. But without giving time enough for the pupils to feel the beauty of it until they had assimilated and were ready for the next step, they were hurried on to the very different and much less interesting business of application. Coming back to the particular from the general, their minds were not prepared for the activities of arithmetic. The desire to dwell on the beauty of the general had to be suppressed, and the mind was therefore muddled in the particular. At once all interest in this region of thought was lost, and a fear even of the word logarithm set up. Not only so, but the moral relation to the teacher was lost. A gulf was placed between them. Confidence was lost.

Whether any ultimate moral damage was done it is difficult to say. Contempt for the subject is probably more prominent in the results than moral obliquity. But I would not venture to say that the damage did not go farther than this. And, in any case, is there any damage to a growing personality greater than that of creating a gulf between it and its teachers?

But this I do feel strongly; that whereas the mathematical studies may and should be an avenue to an increasing reverence for the truth and beauty of the universe, unskilful teaching such as I have suggested may close this avenue finally. In doing this, what happens to the number instinct, which is, in more or less degree, I am convinced in all of us? It seems to me that it is likely to come out in the arithmetical operations of ordinary life. I think now of a boy of energy and some native mathematical ability, to whom school and all connected with learning is a bore, whose time is being wasted in the University, who looks forward as one of the things he might do in after life, to developing a quiet seaside place by the introduction of a dancing palace; he reckons up the present price of land, the cost of erection, the probable numbers of dancers and what they would pay, the profits to be obtained. It is a keen alert mathematical operation. Is the gambling instinct a wrongly-associated impulse? Are the operations of the stock exchange or the money market the same activity merely acting without integration into the moral being? Is the divorcing of mathematics from the aesthetic in school one of the most fruitful sources of the world-desolating non-moral application of one of the most powerful of the intellectual faculties? For the man who has a clear sense of the mathematical possibilities of a situation is, *caeteris paribus*, the master of that situation.

Let us think of the other type of character, the sensitive soul, the one whose human sympathies are awake, to whom the world is one of need. In such an one the checking of the mathematical development will throw the balance more heavily on to the side of sentiment and feeling, will encourage loose and unrelated judgments, and action which does not rest on a quick perception and prevision of consequences. Thus the world is more than ever landed in the power of the men whose judgments are untempered by morals.

Let me conclude with a quotation from the article by M. Poincaré, to which I referred at the outset:

"Is there anything that more needs fighting than dishonesty?—for this is one of the most common and degrading of vices of the natural man. Well! when we have caught the spirit of scientific method, of its scrupulous exactness, the horror of any suppression of fact, when we have grown to dread, as a stain upon our honour, the reproach of having even innocently cooked our results in the slightest degree—when this has become for us part of our professional pride, shall we not carry into all our activities this desire for absolute sincerity, even to the point of no longer understanding what drives men to lie? And is not this the surest means to achieve the most rare and most difficult of all sincerities, that which consists in not deceiving one's self?"

E. CUNNINGHAM.

GLEANINGS FAR AND NEAR.

408. When Tennyson wrote *The Vision of Sin*, Babbage read it. After doing so, it is said that he wrote the following extraordinary letter to the poet:

"In your otherwise beautiful poem, there is a verse which reads:

'Every moment dies a man,
Every moment one is born.'

"It must be manifest that, were this true, the population of the world would be at a standstill. In truth the rate of birth is slightly in excess of that of death. I would suggest that in the next edition of your poem you have it read:

'Every moment dies a man,
Every moment $1\frac{1}{16}$ is born.'

"Strictly speaking, this is not correct. The actual figure is a decimal so long that I cannot get it in the line, but I believe $1\frac{1}{16}$ will be sufficiently accurate for poetry. I am, etc." [Per E. B. Escott.]

409. M. St. Leger, Lieutenant-Colonel of the regiment in which [Buonaparte] was ensign, told me that he never, at that period, had heard of [Buonaparte's] progress in any branch of the mathematics.—Walter Savage Landor, *Imaginary Conversations* (1824), p. 135 (footnote to "Kleber and some French Officers").

410. Darley. "Darley (George) from Dublin, mathematician, considerable actually, and also poet; an amiable, modest, veracious, and intelligent man—much loved here [in London]—though he stammered dreadfully."—From a note by *Thomas Carlyle*, quoted by Mr. R. W. King, *Nineteenth Century*, 1923, p. 524.

"I did not mean that mathematics *inspired* poetry, but only that Science was absolutely necessary for such extravagates as *I* am. Only for this cooling Study I should be out of my reason probably . . . and die like a mad dog foaming." From a letter, Oct. 1827, written to Cary, the translator of Dante, by George Darley.—*Loc. cit.* p. 525.

A QUESTION OF TIME.

By D. B. Mair, M.A.

Let A B C be three travellers going along a straight road, each at constant speed. Let the following three events happen in succession, the passing of C by A at O (Fig. 1), the passing of B by A at V, and the passing of B by C at W. At the meeting of A and C let A set his watch by C's, and at the meeting of A and B let B set his watch by A's. Our question is whether the watches of B and C will agree when they meet at W.

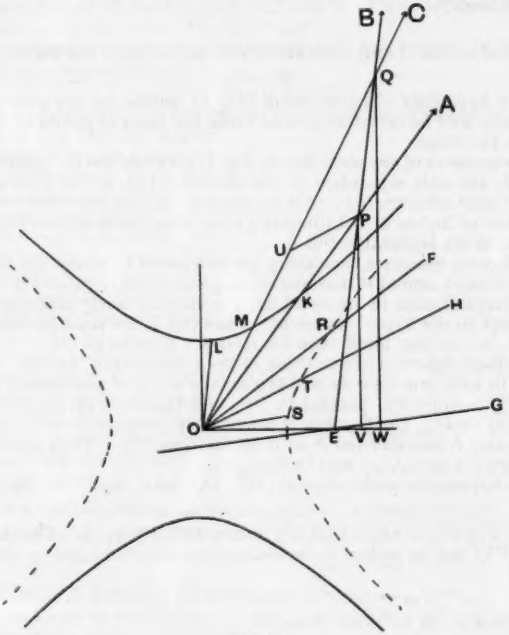

Fig. 1.

On the classical theory there is no doubt that the watches of B and C agree, because there is one unique time for any event, and all true watches show that time for it. But what happens on the relativity theory?

Let D be an observer standing at O where A passes C. Let him take an axis perpendicular to the road OVW along which to measure time. He measures the road in kilometres, and takes 1/300,000 seconds as unit of time, so that a ray of light *in vacuo* travels unit distance in unit time.

At V D draws VP parallel to his time axis and of such length that OV/VP is A's speed relative to him. OP is thus A's world line and P is the location in the fourfold of the event that took place at V on the road, namely the meeting of A and B.

At W D draws WQ parallel to his time axis of such length that OW/WQ is C's speed relative to him. OQ is C's world line and Q is the location in the fourfold of the meeting of B and C at W.

Since B travels at constant speed and P Q are two events on his world line, PQ is his world line.

Any point on the road is at rest for D, and has therefore a world line parallel to D's time axis. The totality of the world lines of points on the road forms a plane, which we may call the world record of the road, or simply the world road. It is the plane of our figure.

Let observer D use coordinates x y, measuring x along the road and y along his time axis. Fig. 1 shows the coordinates of each point with their true values, but it cannot give the true values for slant lines, because the separation σ_{OZ} from the origin to a point Z with coordinates x y is on the relativity theory,

$$\sigma_{OZ}=\sqrt{y^2-x^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The locus of points of unit separation from the origin is the curve

$$y^2-x^2=1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which is a hyperbola when we think of it as defined by the coordinates, but may equally well be called a circle as being the locus of points of unit separation from the origin.

For the measure of any slant line in Fig. 1, we must use its appropriate unit. Thus OM, the unit separation in the direction OQ, is the unit in terms of which we must measure OQ; or if we want to use our foot-rule, we must take the number of inches in OQ divided by the number of inches in OM as the true value of the separation OQ.

Any observer measures time along his own world line, and the road for any observer at any moment is that line of the plane we have called the world road, which is perpendicular to his world line; perpendicularity meaning conjugacy with respect to the hyperbola-circle. Thus OH is the road for observer C at zero time, and at any other time his road is a parallel to OH.

C and other observers like to think of the road as fixed, and for the purpose of falling in with this view we will project the series of parallels on to OH.

From P we draw PU parallel to OH, and therefore perpendicular to OQ, to meet OQ in U. In C's view A travels the distance from U to P in the time OU, and B travels from P to U in the time UQ. Their speeds are thus (in C's view) $v\equiv\sigma_{UP}/\sigma_{OU}$ and $w\equiv\sigma_{UP}/\sigma_{UQ}$.

Let the hyperbolic angle between OP OQ be α, that is to say, let

$$\sigma_{OU}/\sigma_{OP}=\cosh\alpha, \quad \sigma_{UP}/\sigma_{OP}=\sinh\alpha, \quad \sigma_{UP}/\sigma_{OU}=\tanh\alpha. \ldots\ldots\ldots\ldots(3)$$

Let the hyperbolic angle that OQ makes with PQ be β. The shape of the triangle OPQ can be looked on as determined by the angles α β. In fact, we have

$$\sigma_{OQ}=\sigma_{OU}+\sigma_{UQ}=\sigma_{UP}/\tanh\alpha+\sigma_{UP}/\tanh\beta, \quad \ldots\ldots\ldots\ldots(4)$$

so that σ_{UP} is given in terms of σ_{OQ} by

$$\sigma_{UP}=\frac{\sigma_{OQ}}{1/\tanh\alpha+1/\tanh\beta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

while σ_{OP} σ_{PQ} are given by

$$\sigma_{OP}=\sigma_{UP}/\sinh\alpha, \quad \sigma_{PQ}=\sigma_{UP}/\sinh\beta. \ldots\ldots\ldots\ldots\ldots(6)$$

In our problem A was to time the interval from his meeting with C to his meeting with B, and B was to time the interval from this moment to his meeting with C. We want to compare the sum of these times with C's time between his meetings with A and B. A's measure of the time-lapse with which he is concerned is σ_{OP}, the separation between the two events. The measure B was to make is σ_{PQ}. And the sum of these is to be compared with σ_{OQ}, which is C's measure of the time-lapse between his meetings with the others.

Since $\tanh\alpha=\sigma_{UP}/\sigma_{OU}=v$

and $\tanh\beta=\sigma_{UP}/\sigma_{UQ}=w$,

equations 5 and 6 give

$$\sigma_{OP}=\sigma_{UP}\frac{\sqrt{1-v^2}}{v}=\sigma_{UQ}\frac{w\sqrt{1-v^2}}{v+w}, \quad \ldots\ldots(7)$$

$$\sigma_{PQ}=\sigma_{UP}\frac{\sqrt{1-w^2}}{w}=\sigma_{UQ}\frac{v\sqrt{1-w^2}}{v+w}, \quad \ldots\ldots(8)$$

$$\sigma_{OP}+\sigma_{PQ}=\sigma_{UQ}\frac{v\sqrt{1-w^2}+w\sqrt{1-v^2}}{v+w}. \quad \ldots\ldots(9)$$

Thus the estimate of time-lapse made by A and B in co-operation is less than C's value. And at the meeting of B and C, B's watch is found to be slow compared with C's.

Usually the speeds will be small compared with the velocity of light. In this case we have approximately

$$\sigma_{OP}+\sigma_{PQ}=\sigma_{UQ}(1-vw/2). \quad \ldots\ldots(10)$$

We note that although the triangle OPQ is determined in the fourfold by the separation σ_{UQ} and the angles α β, the picture of it that we put on paper may to Euclidean eyes have different appearances. Perpendicularity means conjugacy with respect to the hyperbola-circle. In Euclidean space the hyperbola has a unique pair of conjugate diameters, which are called the principal axes, and are perpendicular to one another. In the hyperbola-circle Euclidean perpendicularity has no meaning, and it is immaterial whether observer D (as in Fig. 1) or observer C (as in Fig. 2) occupies what to our Euclidean eyes is the symmetrical position.

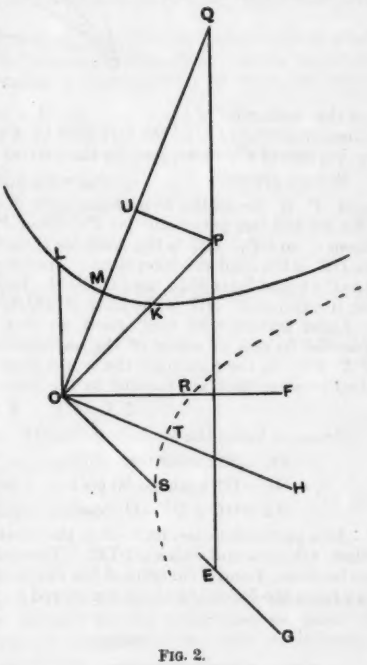

FIG. 2.

Let us suppose that

$$v=w=1-1/20{,}000,$$

so that A and B have speeds a trifle less than that of light. Then

$$\sigma_{OP}+\sigma_{PQ}=\sigma_{UQ}\sqrt{1-v^2}/v=\sigma_{UQ}/100.$$

Thus if A registers 6 months between his meetings with C and B, and B registers 6 months between his two meetings, C (if still alive) will have registered a century between the departure of A and the arrival of B. Or if A turns and comes back, he will have aged by one year only while C has aged by a century.

Relativists used to seek means of avoiding the acceptance of this result. One means was to suggest that it would take A nearly a century to get his vehicle turned round. There can be no doubt about the result; the real difficulty is the means by which A is to perform his somewhat remarkable journey.

In another old illustration observers A C light cigars at the moment of meeting, and each man's cigar lasts 30 minutes, according to his own watch.

One event for our consideration is O (Fig. 3), the place and time where A C meet and light up. Another event is P, the place and time on A's world line

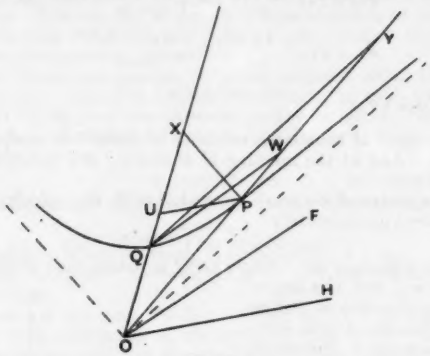

FIG. 3.

of the extinction of his cigar. Another is X on C's world line, the place and time of arrival at C of the last glint of A's cigar. Two more events are Q, the going out of C's cigar, and Y, the arrival at A of the last glint of C's cigar.

We are given $\sigma_{OP}=\sigma_{OQ}=30$ min.,

and P Q lie on the hyperbola-circle of separation 30 min. from the origin. We let fall the perpendicular PU from P on OQ, and the perpendicular QW from Q on OP. OH is the road for C at time zero, and UP, which is parallel to OH, is the road at a later time. Thus events U P happen simultaneously for C at that later time, and OU is the time at which for C A's cigar goes out. Similarly for A OW is the time at which C's cigar goes out.

Light travels with unit speed, so that the world line of a ray of light is parallel to one or other of the asymptotes of the hyperbola-circle. Hence PX QY in the figure are the world lines of the last glints of the two cigars. And because they are parallel to the asymptotes,

$$UX=UP, \quad WY=WQ.$$

Hence, α being the angle between OP and OQ, we have

$$OQ=30 \text{ minutes.}$$

$$OU=OP\cosh\alpha=30\cosh\alpha \text{ minutes.}$$

$$OX=OU+UP=OP(\cosh\alpha+\sinh\alpha)=30(\cosh\alpha+\sinh\alpha) \text{ minutes.}$$

As a particular case, let $\tanh\alpha$ the relative velocity of A and C be 0·866, so that $\cosh\alpha=2$ and $\sinh\alpha=1{\cdot}732$. Then, since the phenomenon is symmetrical as between A and C (in spite of the unsymmetrical appearance of our diagram), we have the following times registered by A and C for the various events.

Events.	Times in minutes from parting as registered by	
	A	C
Parting of A and C	0	0
A's cigar goes out	30	60
C's cigar goes out	60	30
Last glint of A's cigar reaches C	—	112
Last glint of C's cigar reaches A	112	—

D. B. MAIR.

ELEMENTARY METHODS OF TREATMENT OF CIRCULAR, HYPERBOLIC AND ELLIPTIC FUNCTIONS.

By W. Miller, D.Sc., Ph.D.

In the traditional text-book presentation and evaluation of logarithmic, exponential, circular and hyperbolic functions there is a lack of uniformity of method, and the demands on the analytical powers of the average student are considerable, if not excessive It has been shown in *The Mathematical Gazette* of June, 1926, that logarithmic and exponential functions may readily be evaluated by *ab initio* arithmetical methods. Incidentally it may be noted that the theory of the method is independent of the theory of indices. The definition of $\log_e a$ is virtually $\operatorname{Lt}_{x\to 0} \frac{a^x - 1}{x}$, and leads to a simple assignment of *meaning* to irrational and imaginary powers of a number.

It is proposed here to extend the same methods to circular, hyperbolic and elliptic functions (and integrals).

If hyperbolic functions are defined analytically, there is little to add to what has been written under logarithms and exponential functions : to the beginner, however, such a method scarcely suggests Trigonometry or even the name " Hyperbolic "

Circular Functions. It is easily shown geometrically that

$$\cot 2\phi \equiv (\cot^2\phi - 1) \div 2\cot\phi \quad \text{or} \quad \cot\phi(\cot\phi - 2\cot 2\phi) \equiv 1.$$

Multiplying by $1/4^n$, replacing ϕ by $\theta/2^n$, and writing

$$C_n \equiv 1/2^n \cot\theta/2^n,$$

we get

$$C_n(C_n - C_{n-1}) = 1/4^n. \quad \text{.....I}$$

Also $C_\infty = 1/\theta$, θ being in radians.

Equation I is similar to that used in the calculation of logs., and may be written

$$C_n - C_{n-1} = \frac{1}{4^n C_n}. \quad \text{.....II}$$

Summing for all values of n from 1 to ∞,

$$C_\infty - C_0 = \sum_1^n \frac{1}{4^n C_n}.$$

If C_0 is large, we may replace C_n on the right by C_0, so that

$$C_\infty = \frac{1}{\theta} \doteqdot C_0 + \frac{1}{3C_0} \quad \text{.....III}$$

or

$$\theta \doteqdot \frac{3C_0}{3C_0^2 + 1}. \quad \text{.....IV}$$

We may, when C_0 is large, calculate θ by formula IV, or, more accurately, by the solution of a quadratic equation. In the latter case we have the following relation similar (and similarly derived) to one used in connection with logs., viz. :

$1/\theta =$ 1st factor $+ \frac{1}{3}$ of 2nd factor in last equation solved.

Example. Find $\cot^{-1} 7$.

Solving $C_1(C_1 - 7) = \frac{1}{4}$, we find the factors 7·03553 and 3(·01184). Hence

$$1/\theta = 7{\cdot}04737$$

and

$$\theta = {\cdot}141897 \text{ in radians.}$$

By formula IV, a less accurate method, we get

$$\theta = \tfrac{21}{148} = \cdot 141892.$$

Similarly, $\cot^{-1} 13 = \cdot 076772$

and $\cot^{-1} 38 = \cdot 0263098.$

For conversion from radians to degrees we must determine π.

The following equations are easily verified :

$$\begin{aligned}
\cot^{-1} 4\tfrac{1}{2} &= \cot^{-1} 7 + \cot^{-1} 13,\\
\cot^{-1} 3\tfrac{2}{7} &= \cot^{-1} 4\tfrac{1}{2} + \cot^{-1} 13,\\
\cot^{-1} 3 &= \cot^{-1} 3\tfrac{2}{7} + \cot^{-1} 38,\\
\cot^{-1} 2 &= \cot^{-1} 3 + \cot^{-1} 7,\\
\cot^{-1} 1\tfrac{1}{3} &= \cot^{-1} 2 + \cot^{-1} 7,\\
\cot^{-1} 1\tfrac{3}{13} &= \cot^{-1} 1\tfrac{1}{3} + \cot^{-1} 13,\\
\cot^{-1} 1 &= \cot^{-1} 1\tfrac{3}{13} + \cot^{-1} 13 + \cot^{-1} 38.
\end{aligned}$$

Adding, $\pi/4 = \cot^{-1} 1 = 3 \cot^{-1} 7 + 4 \cot^{-1} 13 + 2 \cot^{-1} 38.$

Substituting the values above, we find

$$\pi = 3 \cdot 1416.$$

By adding the above equations in two's, three's, four's, etc., we find

$$\cot^{-1} 4\tfrac{1}{2},\ \cot^{-1} 3\tfrac{2}{7},\ \text{etc.},\quad e.g.\ \cot^{-1} 1\tfrac{1}{3} = 34^\circ\ 41' \cdot 703,$$

the angle being converted to degrees, etc., by means of π.

From the skeleton table formed by such values any details may be filled in as follows :

Example. Find θ, given $\cot \theta = 1 \cdot 3854$.

The nearest tabulated value is $\cot^{-1} 1\tfrac{1}{3}$.

Also
$$\begin{aligned}
\cot^{-1} 1 \cdot 3854 - \cot^{-1} 1\tfrac{1}{3} &= \cot^{-1} 50 \cdot 8288\\
&= 1^\circ\ 7' \cdot 632
\end{aligned}$$

by the above methods.

Hence $\cot^{-1} 1 \cdot 3854 = 35^\circ\ 49'\ 20'' \cdot 1$

with remarkable accuracy, even in seconds.

As in the case of logarithms, such calculations are reversible : *e.g.* cot 10° is easily found to be 5·67127.

Tables of circular functions may easily be calculated, and the principle of proportional parts, derivatives, area of circle, etc., easily established.

The avoidance of unreal quantities is a gain to teaching in the elementary stages, but to show that nothing is thereby lost we proceed to derive a *meaning* for imaginary powers of a number, perhaps the most obscurely treated portion of this region of the work.

Multiplying equation I by $-i^2$,

$$iC_n(iC_{n-1} - iC_n) = 1/4^n.$$

This is of the same form as the logarithmic equation

$$T(T_{n-1} - T_n) = 1/4^n.$$

Hence $T_n = \pm iC_n,$

giving, when $n = \infty$, $\tfrac{1}{2} \log_e N = \mp i\theta,$

and, when $n = 0$, $\dfrac{N+1}{N-1} = \dfrac{\cos\theta}{\mp i \sin\theta}.$

It follows easily that $e^{i\theta} \equiv \cos\theta + i \sin\theta$, *i.e.* a meaning has been found for $e^{i\theta}$.

Hyperbolic Functions. Assuming an elementary knowledge of projections but none of conics, we may define the rectangular hyperbola as follows: P is a pt. on the curve, and p the foot of the $\perp$ar on OX, and c is a constant such that

$$Op \,.\, pP = \tfrac{1}{2}c^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{V}$$

If CD is *any* line, as in Fig. 1, through P, then by similar $\triangle$s,

$$\frac{OC - Op}{pP} = \frac{OC}{OD}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{VI}$$

Multiplying V and VI,

$$Op^2 - OC \,.\, Op + \frac{OC}{OD} \cdot \frac{c^2}{2} = 0,$$

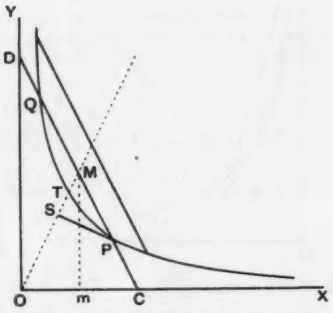

FIG. 1.

showing that there are two positions of P, viz. P and Q, for a given CD, and such that

$$Op + Oq = OC \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{VII}$$

and

$$Op \,.\, Oq = \frac{OC}{OD} \cdot \frac{c^2}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{VIII}$$

By using the relation

$$\frac{OD - Pp}{Op} = \frac{OD}{OC} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{VI}'$$

instead of VI, we get

$$Pp + Qq = OD$$

and

$$Pp \,.\, Qq = \frac{OD}{OC} \cdot \frac{c^2}{2}.$$

For parallel positions of CD, $\dfrac{OC}{OD}$ is unaltered; hence from VIII we have

THEOREM I. *If PQ is one of a set of parallel chords, the product of the projections of OP and OQ on OX (or OY) is constant.*

If M is the mid-pt. of PQ,

$$\frac{Mm}{Om} = \frac{\tfrac{1}{2}(Pp + Qq)}{\tfrac{1}{2}(Op + Oq)} = \frac{OD}{OC},$$

a constant for the parallel chords.

This gives us OM, a "diameter" through O, cutting the curve at T. It is easily seen that the tangent at T is parallel to the chords and has the same inclination to XO that OM has to OX.

If PS is the tangent at P, it follows that $\angle POT$ is equal to the angle between the tangents at P and T or between the tangent at P and PQ.

Hence $\angle MPS = \angle MOP$,

and from similar triangles $OM \cdot MS = MP^2$.

It easily follows that QS is the tangent at Q, and $\triangle SOP = \triangle SOQ$.

If the tangent at T meets PS and QS in U and V, it readily follows that triangles UOP, UOT, VOT, QOV are equal. Continuing in this way we find:

THEOREM II. *Any sector OPQ bounded by OP, OQ and the curve is bisected by the diameter OT bisecting the chord PQ.*

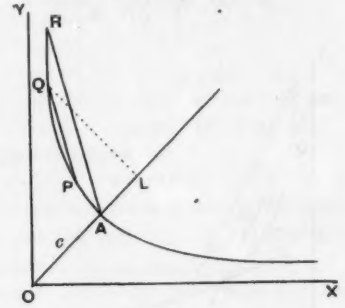

FIG. 2.

If OA, Fig. 2, is the axis of symmetry and AR is drawn parallel to the chord PQ, the chords AR, PQ have a common diameter bisecting sectors OAR and OPQ. Hence sector AOP is equal to sector OQR, and we have a construction for the *sum* of two sectors. If sector areas below the axis OA are reckoned negative, the construction for the *difference* of two sectors will suggest itself.

It is easily seen that $OA = c$ and the area of the square of diagonal OA is $\frac{1}{2}c^2$. If this square is used as a unit of measurement, we may denote the numerical value of the area of sector AOP by μ (*i.e.* sector $AOP = \frac{1}{2}c^2u$). If Q is defined by v, R is defined by $u+v$. Also

$$\sinh v \equiv \frac{QL}{OA}, \quad \cosh v \equiv \frac{OL}{OA}, \quad \coth v \equiv \frac{OL}{QL}.$$

The projection of OQ is equal to the sum of the projections of OL ($c \cosh v$) and LQ ($c \sinh v$).

Applying Theorem I to the chords PQ and AR, projecting (i) on OY, (ii) on OX, we have

$$(\cosh u + \sinh u)(\cosh v + \sinh v) = \cosh(u+v) + \sinh(u+v),$$
$$(\cosh u - \sinh u)(\cosh v - \sinh v) = \cosh(u+v) - \sinh(u+v).$$

By addition and subtraction,

$$\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v,$$
$$\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v,$$

and hence $$\coth u = (\coth^2 u/2 + 1) \div 2 \coth u/2.$$

Now, adopting the method employed for circular functions, we find the equation

$$T_n(T_{n-1} - T_n) = 1/4^n, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{IX}$$

where $$T_n \equiv 1/2^n \coth u/2^n.$$

Equation IX is identical with the logarithmic equation, and it is easily shown that $T_\infty = 1/u$.

$$T_0 = \frac{\cosh u}{\sinh u} = \frac{N+1}{N-1}.$$

Hence $$\sqrt{N} = \cosh u + \sinh u.$$

Also, $$u = \tfrac{1}{2}\log_e N = \log_e(\cosh u + \sinh u)$$

or $$\cosh u + \sinh u = e^u.$$

Similarly, $$\cosh u - \sinh u = e^{-u},$$

so that $$\cosh u = \tfrac{1}{2}(e^u + e^{-u}),$$

$$\sinh u = \tfrac{1}{2}(e^u - e^{-u}).$$

Applications to hyperbolic quadratures follow easily.

While adhering to the geometrical definitions of u, $\sinh u$, $\cosh u$, etc., given above, we may avoid the study of geometrical properties of the hyperbola as follows.

It has been shown that $\dfrac{d}{dx}\log_e x = \dfrac{1}{x}$.

Hence, if a is the foot of the $\perp$ar from A on OX (Fig. 2),

$$\text{area } AapP = \int_x^{\sqrt{2}c} \frac{c^2}{2x}\,dx = \frac{c^2}{2}\log_e \frac{c}{x\sqrt{2}}.$$

But $$x = \frac{c}{\sqrt{2}}(\cosh u - \sinh u);$$

$$\therefore \text{ area } AapP = -\frac{c^2}{2}\log_e(\cosh u - \sinh u).$$

From the equation to the curve $\triangle OPp = \triangle OAa$,

and $$\text{area } AapP = OAP + (\triangle OAa - \triangle OPp)$$
$$= \tfrac{1}{2}c^2 u.$$

Hence $$\log_e(\cosh u - \sinh u) = -u,$$

or $$\cosh u - \sinh u = e^{-u}.$$

Similarly, $$\cosh u + \sinh u = e^u.$$

Hence the relations between exponential and hyperbolic functions.

Elliptic Functions and Integrals. The methods for all previous functions depend on the formation and solution of certain types of quadratic equations. For logarithmic and hyperbolic functions this equation is

$$T_n(T_{n-1} - T_n) = 1/4^n.$$

This may be written $$T_{n-1} = T_n + \frac{1}{4^n T_n}.$$

On squaring and transforming, we obtain

$$T_n^2\left(T_{n-1}^2 - \frac{2}{4^n} - T_n^2\right) = \frac{1}{16^n}.$$

This is again a quadratic equation from which results may be derived of a much higher order of accuracy, but is quoted here only to show the analogy with the equations for elliptic functions.

The first elliptic integral corresponds to θ of circular functions or u of hyperbolic functions. The elliptic functions correspond to the trigonometrical functions above. Corresponding to the trigonometrical relations derived geometrically, similar elliptic-functional relations are easily derived by geo-

metry and lead to the equation corresponding to that for $\tan\theta$ in terms of $\tan\theta/2$. This equation is

$$\operatorname{sn}^2 u = \frac{4\operatorname{sn}^2\frac{u}{2}\left(1-\operatorname{sn}^2\frac{u}{2}\right)\left(1-k^2\operatorname{sn}^2\frac{u}{2}\right)}{\left(1-k^2\operatorname{sn}^4\frac{u}{2}\right)^2}.$$

Put $$\xi(u)\equiv\frac{1}{\operatorname{sn} u}, \quad \eta(u)\equiv\frac{-\operatorname{cn} u\,.\,\operatorname{dn} u}{\operatorname{sn} u}=\frac{d}{du}\log\xi(u)$$

and $$\xi_n\equiv\frac{1}{2^n}\xi\left(\frac{u}{2^n}\right), \quad \eta_n\equiv\frac{1}{2^n}\eta\left(\frac{u}{2^n}\right).$$

The above equation then splits into two, viz. :

$$\xi_n^2\left(\eta_n^2+\frac{1+k^2}{4^n}-\xi_n^2\right)=\frac{k^2}{4^{2n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

$$\eta_n^2\left(\xi_{n-1}^2-\frac{2(1+k^2)}{4^n}-\eta_n^2\right)=\frac{(1-k^2)^2}{4^{2n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B)$$

Beginning from $\xi_0=\dfrac{1}{\sin x}$, where x is the upper limit of the integral, and solving (B) and (A) alternately, we rapidly arrive at a value of $1/u^2$, and therefore of u, the integral.

Complete tables are thus easily obtained for the upper limit, the first and second integrals, $\operatorname{sn} u$ and $\log_{10}\operatorname{sn} u$, as well as for the differences for the last function. Easy exact interpolation formulae fill in all gaps in the table. The results may also be used for the evaluation of the third integral.

No involved study of elliptic functions is required, and, owing to the comparative simplicity of theory and calculations, the method of approach should appeal to students of Physics who may require to use or even calculate such integrals.

It may appear that the solution of a number of such quadratic equations involving irrational numbers would involve much mechanical labour. The equations are however of a type that may be solved by an operation resembling contracted division. These arithmetical methods however depend on a new method of multiplication beginning with digits on the left of both factors.

W MILLER.

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411. *L'Astronomie* (March 1926) quotes from a Parisian daily paper, in which a writer had been exposing astronomical errors in the works of Coleridge, Lamartine, Victor Hugo, de Musset, Rostand, etc. A week later, the writer of the former article commented on the two sunspots just observed by the *Soc. Astron. de France*, each broader than the surface of the Earth :

" Great as these spots are, there is nothing new about them. They are the same spots we saw 11 years ago, and the same as were seen 11 years before that, and so on, for that part of the sun on which they are seen becomes visible to us again after each period of 11 years."

It also quotes from a journal in the north of France :

" The nearest of these stars is some 4 light years from the earth. . . . A detailed list of them is at present being drawn up by Messrs. Hipparchus and Ptolemy. . . . Very clear pictures of the nebulae have been taken at the Observatories of Juvisy and President Wilson."

412. *Mascarille.* Te souvient-il, Vicomte, de cette demi-lune que nous emportâmes sur les ennemis au siège d'Arras ?

Jodelet. Que veux-tu dire avec ta demi-lune ? C'était bien une lune toute-entière.—*Les Précieuses Ridicules.*

HOMOGENEITY APPLIED TO ANGLES.

By Prof. D. M. Y. Sommerville, D.Sc.

In all general equations in mechanics the fundamental units, length, time, mass, enter homogeneously; the result is that these equations are true whatever the units of length, time, and mass may be. Angles, however, occupy an anomalous position. An equation which involves angles, angular velocities or accelerations, is true only for a certain particular unit of angle, usually the radian. Thus the formula $T=2\pi/\omega$, for periodic time with angular velocity ω, is only true when the angular unit is the radian; so also for $\text{arc}=r\theta$, or for $\frac{d}{d\theta}\sin\theta=\cos\theta$. The reason is, that these equations are not homogeneous in angle; π, $\sin\theta$, and $\cos\theta$ are pure numbers, arc and r are lengths. In explanation it is commonly said that angle is of zero dimensions in length, time, and mass. So it is, and yet it is not a pure number, but a quantity just as much as length or any other concrete magnitude. If π were the symbol for a certain angle, instead of being a pure number, equations like $T=2\pi/\omega$, $\alpha+\beta+\gamma=\pi$, etc., would become homogeneous, and true for all angular units. π would then resemble the symbol g, which stands for a particular acceleration, 32·2 ft./sec.^{2} or 981 cm./sec.^{2}, etc.; but π has too long been identified with the number 3·14159..., and is too useful in all sorts of questions not involving angles at all, to be appropriated in this way. If, however, we introduce another symbol, say P, for the angle π radians or the angle of half a revolution, we may make all our equations homogeneous in angles, and restore their generality; it will do equally well, and be generally simpler, if instead of P we use the symbol K, denoting an angle of one radian.

We shall apply this to some of the simple fundamental formulae. Naturally they will become more complicated, for the simplicity of the usual formulae just depends on the particular choice of angular unit.

π is the number 3·14159... and is equal to the ratio of the circumference of a circle to its diameter, *i.e.* circumference $=2\pi r$.

$Arc=a=r\theta$. This equation is really Euclid's proportion

$$\text{arc} : \text{semi-circumference} = \theta : P,$$

hence

$$a=r\theta\pi/P.$$

When $a=r$, $\theta=1$ radian, hence the ratio $P/\pi=1$ radian $=K$, and we have

$$a=r\theta/K.$$

The *trigonometrical functions* are pure numbers. If θ is the angle, we may not therefore use the notation $\sin\theta$, etc., since this is a function of the angle-quantity, and can only be a number when θ is involved always in its ratio to another angle. The most convenient base-angle being of course the radian, we take the ratio θ/K, and then we have the function $\sin(\theta/K)$ or $\sin(\pi\theta/P)$, which may be defined, as usual, either geometrically, if we are content to deal with real numbers, or more generally by the infinite series or the exponential equivalent

$$\sin(\theta/K)=(e^{i\theta/K}-e^{-i\theta/K})/2i,$$

which, of course, is more simply expressed in terms of the number x:

$$\sin x=(e^{ix}-e^{-ix})/2i.$$

Correspondingly, $\sin^{-1}y$ should be defined as a pure number x, such that $\sin x=y$. The corresponding angle is $K\sin^{-1}y$. Beyond the elementary stage, in fact, we do not deal with angles at all in such formulae, but only with pure numbers.

$\lim_{x\to 0} \dfrac{x}{\sin x} = 1$, when x is a pure number, and $\sin x$ is defined as above. But if we use the symbol θ for an angle we must write

$$\lim_{\theta\to 0} \frac{\theta}{\sin(\theta/K)} = K = 1 \text{ radian}.$$

$\dfrac{d}{dx}\sin x = \cos x$ likewise assumes that x is a pure number. For angles we have $\dfrac{d}{d\theta}\sin(\theta/K) = \dfrac{1}{K}\cos(\theta/K)$.

In *circular motion*, angular velocity $= \dfrac{d\theta}{dt} = \dot{\theta}$. Transverse velocity $= r\dot{\theta}/K$, tangential acceleration $= r\ddot{\theta}/K$, normal acceleration $= r\dot{\theta}^2/K^2$.

Simple harmonic motion. If $\dot{\theta} = \omega$, a constant angular velocity, $\theta = \omega t + \alpha$.

$$x = a\cos(\omega t + \alpha)/K,$$
$$\dot{x} = -a\omega/K \sin(\omega t + \alpha)/K,$$
$$\ddot{x} = -(\omega/K)^2 x = -\mu x.$$

μ is not the square of an angular velocity, but is simply a number divided by t^2. The corresponding angular velocity $\omega = K\sqrt{\mu} = P\sqrt{\mu}/\pi$, and the periodic time

$$T = \frac{2P}{\omega} = \frac{2\pi}{\sqrt{\mu}}.$$

Solid angle. Solid angle is a quantity bearing a relation to plane angle analogous to the relation of area to length. The solid angles at the centre of a sphere are proportional to the surface areas which subtend them. If $2Q$ is the whole solid angle at a point, and ω any solid angle subtended at the centre by an area A on the surface of a sphere of radius r,

$$\frac{\omega}{2Q} = \frac{A}{4\pi r^2};$$

therefore $$A = 2\pi r^2 \omega/Q.$$

Let a be the length of the side of a small regular quadrilateral of area A on the surface of the sphere, let a subtend an angle θ at the centre and A subtend a solid angle ω, then $\lim_{a\to 0} A/a^2 = 1$.

Therefore $$\lim_{\theta\to 0} \frac{2\pi r^2 \omega/Q}{r^2\theta^2/K^2} = 1, \quad i.e. \lim_{\theta\to 0} \frac{\omega}{\theta^2} = \frac{Q/2\pi}{K^2}.$$

By making this limit $= 1$, we get a relation between the units of plane angle and solid angle, viz. if the whole plane angle at a point has the numerical value 2π, the whole solid angle has the value 4π ; and generally

$$\frac{Q}{2\pi} = K^2 = (1 \text{ radian})^2,$$

and $$Q = 2\pi(\text{rad.})^2 = 2P^2/\pi.$$

We may therefore call this unit of solid angle the square radian or radian^2. If $\omega = 1$ rad^2., $A = \dfrac{4\pi r^2}{4\pi \text{ rad}^2.} \times 1 \text{ rad}^2. = r^2$, *i.e.* the square radian is the solid angle subtended at the centre by an area equal to the square of the radius.

Since 1 rad. $= \dfrac{180}{\pi}$ degrees, 1 rad^2. $= \left(\dfrac{180}{\pi}\right)^2$ square degrees, and the whole solid angle $= \dfrac{4 \times 180^2}{\pi} = 41252{\cdot}98$ square degrees. Actually a regular spherical quadrilateral, each of whose sides is 1°, is $\dfrac{1}{41250{\cdot}87}$ of the whole surface.

The expression for the area of a spherical triangle becomes

$$\Delta = \pi r^2(\alpha+\beta+\gamma-P)/P = r^2\left(\frac{\alpha+\beta+\gamma}{K}-\pi\right).$$

Dihedral angle. A dihedral angle, though it may be measured by means of a plane angle, is of the dimensions of solid angle.* If a sphere of unit radius is drawn having its centre on the edge of a dihedral angle, the angle is measured by the lune which it forms on the sphere. The whole dihedral angle at an edge is thus $2Q$. If A is the magnitude of the dihedral angle and α that of the plane angle formed on a plane perpendicular to the edge, we have

$$A/Q = \alpha/P, \quad \text{or} \quad A = 2\alpha P/\pi = 2\alpha K.$$

E.g. the angle between two perpendicular planes $= \pi$ rad^2. $= \frac{1}{2}Q$.

The relation between a trihedral solid angle ω and its dihedral angles A, B, C is

$$\omega = \tfrac{1}{2}(A+B+C-Q).$$

Victoria University College,
Wellington, N.Z. D. M. Y. SOMMERVILLE.

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413. *Horace Walpole to Richard West.*

King's College, Cambridge.
August 17, 1736.

... Great mathematicians have been of great use: but the generality of them are quite unconversible; they frequent the stars, *sub pedibusque vident nubes*, but they can't see through them. I tell you what I see: that by living amongst them, I write of nothing else; my letters are all parallelograms, two sides equal to two sides; and every paragraph an axiom, that tells you nothing but what every mortal almost knows. ...

West to Walpole.

MY DEAREST WALPOLE, August, 1736.

Yesterday I received your lively—agreeable—gilt—epistolary—parallelogram, and to-day I am preparing to send you in return as exact a one as my little *compass* can afford you. And so far, sir, I am sure we and our letters bear some resemblance to parallel lines, that, like them, one of our chief properties is, seldom or never to meet. ...—*Correspondence of Gray, Walpole, West and Ashton*, ed. Toynbee, 1915, vol. i. pp. 98-9.

Thomas Gray to Richard West.

Peterhouse, Cambridge,
December, 1736.

... Must I plunge into metaphysics? Alas, I cannot see in the dark; nature has not furnished me with the optics of a cat. Must I pore upon mathematics? Alas, I cannot see in too much light; I am no eagle. It is very possible that two and two make four, but I would not give four farthings to demonstrate this ever so clearly; and if these be the profits of life, give me the amusements of it. The people I behold all around me, it seems, know all this and more, and yet I do not know one of them who inspires me with any ambition of being like him. ...

West to Gray.

Christ Church, Oxford,
December 22, 1736.

... My eyes, such as they are, like yours, are neither metaphysical nor mathematical; I have, nevertheless, a great respect for your connoisseurs that way, but am contented to be their humble admirer. ... —*Ibid.* vol. i. pp. 112-4. [Per F. Puryer White.]

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* For pointing this out I am indebted to Major P. A. MacMahon.

MATHEMATICAL NOTES.

864. [K¹. 21.] *In a given circle to inscribe (by means of the Elliptic Compasses) the Regular Polygons of* 13 *and* 26 *sides.*

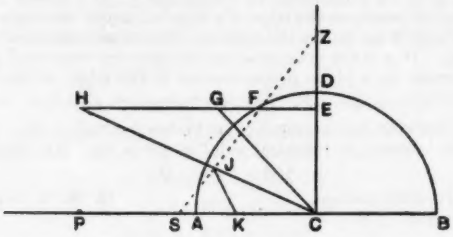

FIG. 1.

Describe a circle ADB (centre C, radius $=1$). (Fig. 1.)
Draw diameter AB, and ordinate DC.

Take a point F on the circumference, so that $A\hat{C}F=60°$.
From F draw FE perpendicular to DC, and produce EF to G, so that $FG=AC/3$; and in AC take K so that $AK=AC/3$.
Join CG, and produce EG to H so that $GH=GC$. Join HC.

Through K draw KJ making $A\hat{K}J=60°$, and cutting HC in J.
Now apply the Elliptic Compasses so that the right-angled frame coincides with ACD, and with the moving limb SZ adjusted to a length equal to twice JC. Slide the limb till it passes through J without being bisected there. Take $SP=HF/2$.*
Now by construction we have obtained that

$$EF=1/2\,;\quad CE=\sqrt{3}/2\,;\quad CK=2/3\,;\quad GF=1/3\,;\quad GC=\sqrt{13}/3\,;$$
$$HE=(5+2\sqrt{13})/6\,;\quad HC=(\sqrt{13}+1)\sqrt{26+2\sqrt{13}}/12\,;$$

and also, because $JC : KC :: HC : HF$, that $JC=\sqrt{26+2\sqrt{13}}/6$.

Also we have that $\cos H\hat{C}B=-HE/HC=-(7+\sqrt{13})/2\sqrt{26+2\sqrt{13}}$, and that $J\hat{S}C=H\hat{C}B/3$; so that calling $2\cos J\hat{S}C=f$, then

$$2\cos H\hat{C}B=f^3-3f=-(7+\sqrt{13})/\sqrt{26+2\sqrt{13}},$$

and
$$CP=(f\sqrt{26+2\sqrt{13}}+\sqrt{13}+1)/6=S_1.$$

From S_1 we proceed to define other values: S_2, S_3, S_4, S_5 and S_6.
Thus

$$S_2=S_1^2-2=[f^2(26+2\sqrt{13})+2f(1+\sqrt{13})\sqrt{26+2\sqrt{13}}-58+2\sqrt{13}]/36,$$
$$S_4=S_2^2-2=[f^2(26+14\sqrt{13})+2f(4+\sqrt{13})\sqrt{26+2\sqrt{13}}-58-34\sqrt{13}]/36,$$

when reduced by means of the equation

$$f^3-3f=-(7+\sqrt{13})\sqrt{26+2\sqrt{13}}.$$

Similarly, $\quad S_5=2-S_4^2\,;\quad S_3=2-S_5^2\,;\quad S_6=S_3^2-2\,;$

and ultimately we find that $S_1=2-S_6^2$.

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* *Vide* article *Mathematical Gazette*, Oct. 1924.

So that we have the abscissa S_1 ($=CP$) connected with our unit radius by a system of six equations (written most conveniently for the geometer):

$S_1^2-1^2=1(1+S_2)$, ...(I) | $1^2-S_5^2=1(S_3-1)$, ...(IV)

$S_2^2-1^2=1(1+S_4)$, ...(II) | $S_3^2-1^2=1(1+S_6)$, ...(V)

$S_4^2-1^2=1(1-S_5)$, ...(III) | $1^2-S_6^2=1(S_1-1)$. .....(VI)

Now (Fig. 2) taking the pt. Q in the circumference so that $PQ=AC$, we join PQ, QC, and with centre Q describe a circle through P and C. Join QA and produce to Y. Produce PQ to R. Join RC and produce to T. Join TQ, cutting circle ADB in V. Join CV and produce to W. Join WQ and produce to cut circle ADB in X.

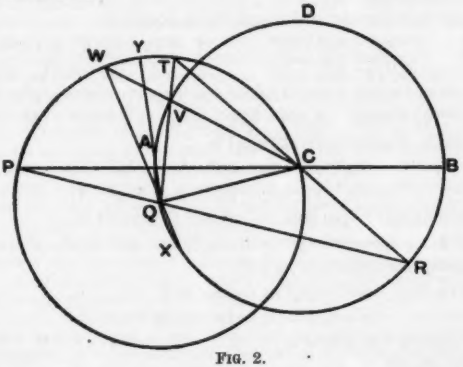

FIG. 2.

Now because $PC=S_1$, we have by (I) that $QR=S_2$; by (II) that $CT=S_4$; by (III) that $QV=S_5$; by (IV) that $WC=S_3$; by (V) that $QX=S_6$; and also by (VI) that $QA=S_6$; therefore $QA=QX$,

and $$A\hat{Q}C=C\hat{Q}X=2\,.\,W\hat{C}Q, \quad \text{.....(VII)}$$

and $$Y\hat{Q}W=A\hat{C}Q. \quad \text{.....(VIII)}$$

Again, $Q\hat{C}T=2\,.\,C\hat{Q}R=4\,.\,A\hat{C}Q$; and therefore $P\hat{C}T=3\,.\,A\hat{C}Q$, and $P\hat{Q}T=6\,.\,A\hat{C}Q$.

But $P\hat{Q}W=2\,.\,P\hat{C}W=4\,.\,Y\hat{Q}T$, so that $P\hat{Q}T=5\,.\,Y\hat{Q}T+Y\hat{Q}W=6\,.\,A\hat{C}Q$; or, by (VIII), $5\,.\,Y\hat{Q}T=5\,.\,A\hat{C}Q$, and $Y\hat{Q}T=A\hat{C}Q$.

Thus $P\hat{C}W=2\,.\,A\hat{C}Q$, and $W\hat{C}Q=3\,.\,A\hat{C}Q$; and by (VII) $A\hat{Q}C=6\,.\,A\hat{C}Q$.

Thus we have the isosceles triangle CAQ with each triangle at the base six times that at the vertex; and AQ and QX will be sides of the regular inscribed polygon of 26 sides; and AX (if joined) will be a side of the regular inscribed polygon of 13 sides.

C. H. CHEPMELL.

69 Wilbury Crescent, Hove, Sussex.

865. [L¹. 6. a.] *Circle through three Points of a Conic: Circle of Curvature.*

1. *The parabola* $y^2=4ax$.

One circle through the points (x_1, y_1), (x_2, y_2) is

$$C\equiv(x-x_1)(x-x_2)+y^2-4ax=0,$$

and the line joining them is $L\equiv(y-y_1)(y-y_2)-(y^2-4ax)=0$.

Hence the family of circles through the points is $C=\lambda L$, and if one of them goes through the pt. (x_3, y_3) we have

$$(x_3-x_1)(x_3-x_2)=\lambda(y_3-y_1)(y_3-y_3) \quad \text{or} \quad (y_3+y_1)(y_3+y_2)=16a^2\lambda.$$

Hence the equation of the circle through the three pts. can at once be written down.

For the circle of curvature at $(at^2, 2at)$ put $y_3=y_2=y_1\equiv 2at$ in the above;

$$\therefore\ \lambda=t^2,$$

and the required circle is

$$(x-at^2)^2+y^2-4ax=t^2\{(y-2at)^2-y^2+4ax\}, \text{ etc.}$$

2. *The conic* $ax^2+by^2=1$.

In the same way the circle through three points is

$$\frac{(a-b)(x-x_1)(x-x_2)-(ax^2+by^2-1)}{(x_3-x_1)(x_3-x_2)}=\frac{(a-b)(y-y_1)(y-y_2)+(ax^2+by^2-1)}{(y_3-y_1)(y_3-y_2)}$$

or

$$a^2(x_3+x_1)(x_3+x_2)\{(a-b)(x-x_1)(x-x_2)-(ax^2+by^2-1)\}$$
$$=b^2(y_3+y_1)(y_3+y_2)\{(a-b)(y-y_1)(y-y_3)+(ax^2+by^2-1)\},$$

whence the circle of curvature at (x_1y_1) is

$$a^2x_1^2\{(a-b)(x-x_1)^2-(ax^2+by^2-1)\}$$
$$=b^2y_1^2\{(a-b)(y-y_1)^2+(ax^2+by^2-1)\}.$$

For the rectangular hyperbola $x^2-y^2=a^2$ the result is

$$x_1^2\{2(x-x_1)^2-(x^2-y^2-a^2)\}=y_1^2\{2(y-y_1)^2+(x^2-y^2-a^2)\}.$$

3. *The rectangular hyperbola* $xy=c^2$.

The circle on (x_1y_1) and (x_2y_2) as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

and the line joining the points is $(x-x_1)(y-y_2)-(xy-c^2)=0$. Thus the circle through three points is

$$\frac{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)}=\frac{(x-x_1)(y-y_2)-(xy-c^2)}{(x_3-x_1)(y_3-y_2)};$$

or, in parameters,

$$\frac{(x-ct_1)(x-ct_2)+\left(y-\frac{c}{t_1}\right)\left(y-\frac{c}{t_2}\right)}{1+\frac{1}{t_3^2t_2t_1}}=\frac{(x-ct_1)\left(y-\frac{c}{t_2}\right)-(xy-c^2)}{-\frac{1}{t_2t_3}},$$

The circle of curvature at "t" is

$$(x-ct)^2+\left(y-\frac{c}{t}\right)^2=c\left(t^2+\frac{1}{t^2}\right)\left(\frac{x}{t}+yt-2c\right), \text{ etc.}$$

4. *The general conic.*

In the same way, the equation of the circle through three points is

$$\frac{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)}$$
$$=\frac{\text{Secant}}{a(x_3-x_1)(x_3-x_2)+h[(x_3-x_1)(y_3-y_2)+(x_3-x_2)(y_3-y_1)]+b(y_3-y_1)(y_3-y_2)}.$$

The circle of curvature at (x_1y_1) is, since $\frac{dy}{dx}=-\frac{X}{Y}$,

$$\frac{(x-x_1)^2+(y-y_1)^2}{X_1^2+Y_1^2}=\frac{2(xX_1+yY_1+Z_1)}{bX_1^2-2hX_1Y_1+aY_1^2};$$

and since $bX_1^2-2bX_1Y_1+aY_1^2\equiv CS_1-\Delta$, the equation becomes

$$\frac{(x-x_1)^2+(y-y_1)^2}{X_1^2+Y_1^2}=\frac{2(xX_1+yY_1+Z_1)}{\Delta}.$$

N. M. GIBBINS

REVIEWS.

Analytic Functions of a Complex Variable. (Carus Mathematical Monographs, No. 2.) By D. R. Curtiss. Pp. 173. 2$. 1926. (Open Court Publishing Co.)

This is an attractive little book, and it should be of very distinct value to the student who knows a little about complex variables, and who desires to obtain a considerably more extended knowledge of the broad outlines and main trend of the subject without a close investigation of analytical details and formal proofs.

The object of the series of monographs, of which this is the second, is to produce expositions comprehensible not only to teachers and students specialising in mathematics, but also to scientific workers in other fields; and, in the reviewer's opinion, Professor Curtiss has been as successful in this aim as the nature of the subject permits. But when he states in the preface that "Without such knowledge [*i.e.* of elementary differential and integral calculus] one may, however, obtain some idea of the scope and purposes of the theory of functions from the following pages," it is difficult to avoid thinking that he is unduly optimistic. In fact, even the mathematical student may find that the style is, perhaps, a shade too concise. But the reviewer may have formed this opinion in consequence of his preconceived idea that the exposition of the fundamentals of the subject is more effectively given, when possible, in the informal atmosphere of the lecture room by a discursive lecturer than through the formal medium of the printed page; and he would feel no qualms about putting it into the hands of the average boy who has just won an entrance scholarship at Cambridge, to give him something to think about before going up to the university.

The book begins with an explanation of what a complex number is, followed by a digression on functions of real variables, extending as far as a description of Green's theorem, and proceeds to a discussion of derivates with illustrations drawn from the exponential and trigonometrical functions. The author is wisely silent as to what precisely is meant by a "curve" or a "region," and is perhaps a little indiscreet in mentioning the concept of uniformity of continuity, especially when the only subsequent reference to it is the mere statement that it plays an important part in establishing the existence of the definite integral of a continuous function.

After an account of conformal transformation and its physical applications (*e.g.* to two-dimensional fluid motion) we reach the kernel of the book, namely Cauchy's theorem and its proof, which the author bases on Green's theorem; a few pages on convergence of series lead up to the expansions due to Taylor and to Laurent, and the book concludes with an account of poles, essential singularities, analytic continuation and many-valued functions, which makes the mathematical reader wish that the author had gone further into the subject instead of giving appropriate references to various standard treatises.

Elements of the Mathematical Theory of Limits. By J. G. Leathem. Pp. viii + 288. 14s. 1926. (G. Bell & Sons.)

The chief impression which this book conveys to the reader is a sense of the loss which Mathematical Science has sustained by the premature death of its author; for the late Dr. Leathem was a very remarkable man. It is to be presumed that his main interests lay in Applied Mathematics rather than in Pure Mathematics; at least that is the impression which is conveyed by his writings on Hydrodynamics and his well-known *Tract* on Surface and Volume Integrals. But he possessed the quality which is not so common among Applied Mathematicians of being deeply interested in the logic of the analytical processes which lie at the foundations of various branches both of Pure and of Applied Mathematics, and of being able to explain these processes in straightforward untechnical language; this characteristic is evident from a study of his *Tract* and of the work now under review.

This work is eminently readable, and it will doubtless be of great value to a large class of students. Its defects may be due to its lacking the final

revision of the author; for those defects which struck me most could hardly have been removed by a conscientious editor without his feeling that he was exceeding his powers. A professional analyst, on reading the book, would notice at once that, after about a couple of pages in the first chapter, mainly about the significance of extracting the square root of 2, nothing is seriously said about irrational numbers and Dedekind's theory until the last of the thirty-seven chapters in the book; though, in fairness to the author, it must be stated that he is not unique in having adopted such an arrangement, and also that Chapter XXII. contains the essentials of Heine's definition of an irrational number by means of ascending and descending sequences. And many readers with analytical minds would probably feel (as I felt) that the book is unduly prolix, and would easily bear compression to about two-thirds of its present length; in fact, the book in its present form is better fitted to be used as a lucid and moderately informal exposition in a lecture room rather than to be compared with the stronger meat to be found in Hardy's *Course of Pure Mathematics*.

But there exists a class of reader who will find the book of extreme use, namely those who wish to discover what elementary analysis is about, but who have no intention of becoming professional analysts. There is no denying that there exists a large class of persons who are deeply interested in mathematics and who, at the same time, find the usual array of symbols of analysis, δ's and ϵ's, and so forth, extremely repulsive; such readers will find the easy, conversational style to be attractive, and the author to be apparently sympathetic to their difficulties. And they will learn far more about elementary analysis from this book than they would have learned from a book constructed on more severely logical lines and written with the terseness in which many analysts take pleasure.

The book may be divided into three parts, whose lengths are roughly equal; the first may be described as introductory, dealing with general ideas about functions and limits, and the fundamental properties of limits; the second is concerned mainly with methods of approach to limits and approximations; and the third part is the application of the earlier parts of the book to the convergence of series and products. Throughout the book, the illustrative examples are happily chosen to illuminate the theory.

Dr. Leathem has avoided introducing new symbols as far as possible, but he has felt obliged to use the symbol $(=)$ for approximate equality. I cannot fathom his objection to the use of $\sim$ in this sense, which seems to me to have come to stay. His inclined arrows for progress up to or down to a limit are a refinement of the arrow notation which he invented twenty years ago, and they have certain advantages over the arrows with single barbs which are used by some writers. G. N. W.

Modern Algebraic Theories. By L. E. Dickson, Ph.D. Pp. 273, with Index. 15s. 1926. (Sanborn & Co., Chicago.)

This book is an introductory course on the higher branches of algebra, dealing with a wide range of topics, including linear transformations, invariants, matrices, bilinear forms, quadratic forms, elementary divisors, substitution groups, fields of rationality, finite groups, the quintic equation, and the icosahedron. It is somewhat remarkable for an author to be able to cover such a vast region of pure mathematics in a comparatively small book; and undoubtedly this would be impossible to do satisfactorily by anyone who was not a master of his subject. The author is one of the most eminent of American mathematicians, who has practically created a new branch of higher algebra in his modular invariant theory. He has moreover helped to make Chicago a very active centre of mathematical research. The present book is in fact the outcome of lectures to students who have some algebraic aim in view. It brings together, in a convenient shape, the principal facts, ideas and theorems which are needed by anyone who wishes to get a proper perspective in algebra.

Certain fundamental concepts had already been covered by the author's *Algebras and Their Arithmetics*. These are omitted in the present work. So also are the symbolic methods of dealing with quantics and invariants. Inas-

much as these methods are intimately connected with differential operators on the one hand, and provide a ready application of the principles involved in the Frobenius theory of group characters on the other, it seems a pity that they should have been excluded from the book. For both differential operators and group characters are dealt with in the course of the work.

The book is thoroughly well filled with references and suggestions for further reading; and as it is written clearly and simply, it is one that is likely to prove very useful. The author has done a specially good service, incidentally, in providing a stepping stone to a proper understanding of that great book on the Icosahedron by Felix Klein. He has also much that is interesting to say about ruler and compass constructions and finite groups.

The book will repay careful study, and can be warmly recommended.

H. W. TURNBULL.

(I.) **The Theory of Measurements.** By L. TUTTLE and J. SATTERLY. Pp. xi + 333. 12s. 6d. net. 1925. (Longmans, Green & Co.)

(II.) **A First Course in Statistical Method.** By G. I. GAVETT. Pp. viii + 358. 17s. 6d. 1925. (M'Graw-Hill Book Co.)

(I.) This is a book of elementary practical mathematics for science students. Though not called a second edition, it actually owes much of its matter and its general arrangement to a smaller book by the first-named author bearing the same title and published in 1916. Some indication of the mathematical equipment of the student approaching the book is given by a statement in the preface that "no knowledge of trigonometry is presupposed, and none is imposed upon the reader." The subjects range from elementary arithmetic (abridged division and multiplication) to deviation and dispersion, least squares and correlation. It may be doubted whether students with so slender an equipment, presupposed or imposed, are qualified to approach the later subjects.

A commendable feature of the book is its attention to arithmetical detail, the numerous hints on setting out work, shortening labour and ensuring accuracy being valuable. It is all the more regrettable, therefore, that our faith should be undermined by such statements as, for example, in the first paragraph of the chapter on "Accuracy," that 1 yard = 83·8 cm., or that 2·539978 cm. may be taken as 2·540 cm. with an accuracy of one in a million, or various other lapses. Again, some of the diagrams have been hastily drawn (the upper logarithmic scale of Fig. 17, p. 60, is rather warped), so much so that on p. 84 it has been necessary to add a footnote excusing the drawing of the curve $y = e^{-x^2}$, which shows a maximum ordinate perceptibly less than unity.

An innovation singled out for mention in the preface, but persisted in for only one chapter, is the use of the term "dispersion" in the sense of what is usually called "probable error." It may readily be admitted that the latter name is apt to mislead beginners, but the alternative suggestion is not happy, since "dispersion" is already in wide use in the sense of "standard deviation."

The defect of the book seems to be that in the attempt to include mention of as many topics as possible ("Mathematical Theory of Wounds," "Law of Fatigue," etc., etc.), the separate topics themselves receive too scrappy a treatment. The copious tables of the appendix and the profusion of examples in the text are insufficient compensation.

(II.) Mr. Gavett's *Course* has a range of subject-matter which in many places overlaps that of (I.). It follows a different and more successful plan. The discussion of logarithms, permutations, probability, derivatives and integrals is relegated to several appendices, and thus there are no digressions to interrupt the course of the book proper. The author has pruned his material, achieving his purpose with distinct economy by making the fullest use of a few well-chosen recurring examples. The style is direct and fresh, the diction slightly tinged with transatlantic idiom. (Purists will wince at such hybrids as "scatteration.") The printing and diagrams are good, and we noticed only a few errors. The need for the tables at the end of the book is not evident. Detached books of tables are in abundance nowadays, and are much more convenient in practice.

It is probable that to read to any purpose in mathematical statistics the student should approach the subject with a rather better grounding in elementary pure mathematics than is assumed here, but if the limitation be accepted, or if it be unavoidable, then (II.) seems to fulfil its object with a distinct measure of success.

Statistical Analysis. By E. E. Day. Pp. xxvii + 459. 17s. net. 1925. (The Macmillan Co.)

This book differs from the two preceding in being more a general introduction to statistical method than a practical course with exercises. It sets out to survey the various kinds of distributions and series which arise in statistics, and describes in elementary language the processes employed in elucidating them. The principal sections treat of frequency distributions, spatial series, time series, graphic representation and index numbers. Brief but interesting accounts of the collection and tabulation of data are given in appendices.

On the mathematical side there is very little ground for criticism, for the good reason that "the mathematical requirements have been reduced to a minimum." This reduction of mathematical requirements seems to us to reduce the value of the book considerably. It is significant that whereas the preface of the 1st edition of Bowley's well-known "Elements" assumed an equipment rather more moderate than that of a certain Major-General, the 4th edition expressed the opinion "that a student will save time in the end by becoming familiar with the infinitesimal calculus before he commences the serious study of mathematical statistics."

The arrangement and exposition of subjects are both very good, but the deliberate abstention from mathematical formulae has the inevitable result of lengthiness. There is also occasional repetition, and too much indulgence in long quotations. The author's own opinions are advanced with cautious reservations.

The external features of the book are wholly admirable. The printing is excellent, the maps, tables and diagrams could not be set out better, and in all such details of form there is evidence of most careful preparation. It is difficult, however, to see any reason for the inclusion of the thirty pages of tables at the end. The book is quite long enough without them, indeed, for an introduction, too long. So protracted a survey of the surface of the statistical waters makes one disinclined to take the plunge. A. C. Aitken.

Elements of Mathematics for Students of Economics and Statistics. D. C. Jones and G. W. Daniels. University Press of Liverpool. Pp. viii + 240. 8s. 6d. net. 1926. (Hodder and Stoughton.)

"This book is designed for students who have only a slender training in mathematics, and who, in consequence, are afraid of symbols and statistics." In the hands of a good teacher the book should certainly help to provide students with a working knowledge of some of the mathematics that enters, or can be made to enter, into the study of economics. The student who uses the book for private study may find some parts difficult, especially in Part IV., where calculus is "applied to analysis of economic theory." And the solitary student would have been more adequately provided for if there had been more Examples for him to work. What they have done, the authors of this book have for the most part done well; but they have not done enough: study of the book by many students will have to be supplemented by reference to other books; and the authors have not done enough for eight-and-sixpence! The treatment of logarithms, of averages, and the elements of calculus, is clear; the treatment of limits is clear and sound. The explanation of the relation between curves and their derived curves would have been better if the diagram showing the derived curve had been placed immediately beneath the primary curve. The Normal Curve of Error receives a scanty and unconvincing discussion. And one would have liked a chapter on the simple algebra that must enter into any explanation of correlation coefficients. In Part IV., in the course of a discussion of a point in economic theory, the meaning of a differential coefficient, which has already been dealt with in a

previous chapter, is explained at length again ; the authors had already made the idea clear, but we congratulate them on making it clear twice.

Students of economics use mathematics for two quite distinct purposes : as an instrument for the manipulation of statistics, and as a means for expressing certain quantitative ideas such as those of marginal utility, or elasticity of demand. It is the latter use of mathematics that the authors of this book seem to have had chiefly in mind. Not all students of economics find mathematics useful to them in this second capacity ; some feel that the mathematical statement of economic relationships is more precise than the relationships themselves, and that, as Dr. Marshall wrote, " the Mecca of the economist lies in economic biology rather than in economic dynamics." Such students will not find this book relevant to their thinking. They will not feel, for example, that the utility of a symphony to the hearer at a particular instant can be expressed mathematically, or that a graph makes clearer to them the fact that a bicycle ride begins to pall if it goes on too long. But other students will find mathematics, apart from statistical theory, of much service in their attempt to understand the concepts used in economic theory. To quote Dr. Marshall again : " The chief use of pure mathematics in economic questions seems to be in helping a person to write down quickly, shortly and exactly, some of his thoughts for his own use." The shorthand that they require is very clearly explained to such students in this book, and we strongly recommend it to them. For those who think that mathematics plays a more legitimate part in economic statistics than it does in the " pure theory " of economics this book might have catered better. E. R. HAMILTON.

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414. Small need then to say as a wind up that arithmetic and algebra in their wonted setting forth cannot but be educationally bad and mischievous scientifically misleading bewildering unhelping balking stunning deadening and killing and philosophically worthless.—Preface to *Pelicotetics*, A. Sandeman, 1867 [per E. H. Neville].

415. The conditions of a good definition are, (1) perfect axiomatic evidence that the property which is made the distinguishing test belongs to the object intended to be defined, and to nothing else; (2) entire separation of the part of the property, if any, which admits of being demonstrably connected with the notion defined, from that which contains the assumption; (3) the introduction of the definition in the proper place, namely, when the necessity for a new verbal representation has begun to appear.—De Morgan, " Definition," *Penny Cyclopaedia.*

416. One of the earliest things I remember is the discovery of the difference between even and odd numbers. . . . I soon got to do the most difficult sums, always in my head, for I know nothing of figures beyond numeration. . . . I did these sums much quicker than any one could on paper, and I never remember committing the smallest error.—Archbishop Whately.

417. Toast. To the health of the prime numbers, the only branch of mathematics which has not been defiled by contact with the concrete.

418. I have been cooking myself dry, and am now in my nightgown ; and this moment comes a Letter to me from one Wheldon who tells me that he hears I am a lover of the Mathematicks, that he has found out the Longitude, shown his discussion to Dr. Dobbs of y^e^ College, and sent letters to all the Mathematicians in London 3 months ago, but received no answer, and desires I would read his discourse. I sent back his Letter with my answer under it, too long to tell you, only I said I had too much of the Longitude already, by 2 projectors, whom I encouraged, one of whom was a cheat and the othr cut his own throat, and for himself I thought he had a mind to deceive others, or was deceived himself. And so I wait for dinner.—Swift's *Journal*, 1727, Sept. 26.

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ERRATA.

In the last list (vol. 13, p. 216) under N. H. ABEL, for 'Ostwald III.' *read* 'Ostwald 111'.

The author of Note 860, p. 250, was Mr. A. Dakin, M.A. We hope that both gentlemen will accept apologies for the error.

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